Existence, Stability, And Boundedness in Second-Order Difference Systems with Exponential Components

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Abstract

This work studies the existence, stability, and boundedness of solutions for second-order difference systems involving exponential components. Difference systems provide a useful framework for describing the behavior of discrete-time models arising in applied mathematics, economics, biology, engineering, and other areas. The presence of exponential terms can significantly influence the long-term behavior of solutions and may lead to different types of growth or decay. The main purpose of this study is to examine the conditions under which solutions of the considered systems exist and remain well behaved as the discrete variable increases. Particular attention is given to stability, which describes whether solutions remain close to their initial states or equilibrium solutions, and boundedness, which ensures that the solutions do not grow without limit. By using suitable mathematical techniques and appropriate assumptions on the system parameters, sufficient conditions for these properties are established. The results provide a general understanding of how the coefficients and exponential components affect the behavior of solutions. The study also demonstrates that suitable restrictions on the system parameters can prevent unbounded growth and guarantee stable behavior. These findings contribute to the qualitative analysis of second-order discrete systems and provide a basis for further investigations of more general nonlinear and exponential difference equations.

Keywords: *Second-Order, Parameters, Components, Model, Equations.*

I. INTRODUCTION

Second-order nonlinear differential equations with damping are an essential tool for modelling physical problems that arise in different fields of endeavour, such as biology, chemistry, physics, mechanics, electronics, engineering, economics, atomic energy, and communication theory. Nonlinear differential equations are used to model damped and driven oscillators, prediction of earthquake occurrence, modelling of the brain, and crash analysis. Many authors have proposed various methods in the literature discussing the qualitative behavior of solutions of nonlinear differential equations. In this area, mention can be made of the implicit function theorem, which confirms the existence of a unique solution, and the Lyapunov direct method, which involves the construction of a suitable positive definite function where the derivative of the positive definite quadratic form is negative definite. The qualitative behaviors of solutions of differential equations of second order have been discussed extensively and are still receiving attention from researchers because of their practical applications to natural phenomena. The existence of solutions to ordinary differential equations is very much connected with the fixed point theorem. A number of cases arise in the literature where fixed point theorems have been used to generalize cases of nonlinear differential equations; see for instance the works. On the use of the Leray–Schauder fixed point and integrated equation as a means to establish a priori bounds for nonlinear differential equations see. One of the goals of the search for bounds of solutions to any differential equation is to achieve a global optimum of the solution in a closed and

bounded domain. Many results on the boundedness of solutions of Duffing's equation have been obtained; for instance, see. Author proved that each solution of the particular Duffing's equation of the form:

$$\ddot{x} + 2x^3 = p(t) \quad (1)$$

is bounded for $t \in \mathbb{R}$. Researcher employed the implicit function theorem and Lyapunov functions to establish the existence and stability of periodic solutions for a second-order nonlinear differential equation. Furthermore, Torres (2003) discussed the existence of one-signed periodic solutions of second-order differential equations. Unfortunately, the problems of existence, uniqueness, boundedness and asymptotic stability of periodic solutions of second-order nonlinear differential equations with damping and resonance effects are yet to be investigated. The aim of this paper is to extend the work found in. Consider

$$\ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 = p(t) \quad (2)$$

where a, b, c are real constants and $p : [0,2\pi] \rightarrow R^n$ is continuous with

$$x(0) = x(2\pi)$$

$$\dot{x}(0) = \dot{x}(2\pi)$$

as periodic initial conditions. Here c is the damping coefficient and a is the resonance coefficient, and the dots indicate differentiation with respect to the independent variable t. Equation (2) is equivalent to the system of first-order differential equations

$$\dot{x} = y$$

$$\dot{y} = p(t) - cy - ax - bx^2 - 2x^3$$

where $p(t)$ is periodic, i.e. $[0,2\pi] \rightarrow \mathbb{R}^+$ with $x(t + T) = x(t)$, and $a > 0$, $c > 0$ are constants on which appropriate conditions will be placed in order to solve equation (2). This work is motivated by study of the work.

II. REVIEW OF LITERATURE

Pinelas, Sandra et al., (2024). An important aspect of qualitatively studying delay differential equations is determining if solutions exist and whether they are stable. Numerous branches of engineering and the natural sciences make use of stabilisation by means of appropriately applied impulse controls. Solutions to second-order, multiple-delay impulsive differential equations may be found and stabilised exponentially under the circumstances given in this article. Fixed impulse effects, Schaefer fixed point theorem, and Lyapunov-Krasovskii functionals are the primary techniques discussed in this article.

Ganesan, Arthi et al., (2023). In this study, the instability method, stochastic analysis, and stability theory are used to solve exponential stability conditions for neutral second-order stochastic systems with impulses and state-dependent delay. While the exponential stability of such systems is known to be true in the deterministic situation, it is less well-established in the stochastic case. In this work, we provide the necessary conditions to prove it. A symmetric Wiener process best describes the noise impact in our scenario. An exponential stability analysis of the pth moment of a second-order system subject to stochastic perturbation is developed using the formulation of the impulsive integral approach. A practical example is given to demonstrate the theoretical concept.

Eze, Everestus et al., (2023). Solving certain second-order nonlinear differential equations including damping and resonance effects is the focus of this study. The solutions are examined qualitatively. The stability of the system was investigated by using a full Lyapunov function and Lyapunov's direct technique.

The existence and boundedness of a unique solution were guaranty by establishing adequate criteria using Kwasniewski's fixed point theorem. The findings demonstrate that the point of equilibrium was asymptotically stable. In addition, the use of the Bendix son criteria for a periodicity test confirmed that the second-order nonlinear differential equation's solution is aperiodic, building on previous findings in the literature.

Ibrahim, Tarek et al., (2022). We study the boundedness and local and global asymptotic behaviour of the solutions to an exponential type second-order difference formula ξn+1=a+bξn−1+cξn−1e−ρξn, where a, c, and ρ are real numbers with values between 0 and infinity, b is a real number with values between 0 and 1, and the initials ξ0 and ξ−1 are non-negative real numbers. There are a few more extreme examples provided. To back up the theoretical findings, we provide two real-world numerical instances.

Leiva, Hugo. (2001) Here we examine an abstract second-order differential equation with dissipation in a Hilbert space H: u +cu′+dA u+kG(u) = P(t), u ∈ H, t ∈, where c, d and k are positive constants, G:H→H is a Lipschitz Ian function and P:→H is a continuous and bounded function. An unbounded linear operator A:D(A) ⊂ H→H has compact resolvent, is self-adjoint, and is positive definite. Our proof is that for certain d, c, and k values, this system has a limited solution that is exponentially asymptotically stable under these circumstances. If P(t) is nearly periodic, then this limited solution is almost periodic as well.

$$x_{n+1} = ax_n + by_n{}^{e-y_{n-1}}$$

$$y_{n+1} = cy_n + dx_n{}^{e-x_{n-1}}$$

In this section we analyse the model

$$x_{n+1} = ax_n + by_n{}^{e-y_{n-1}},$$

$$y_{n+1} = cy_n + dx_n{}^{e-x_{n-1}}, \quad n = 0, 1, 2, \ldots \ldots \qquad (3)$$

For which the starting conditions are positive real integers and a, b, c, and d are also positive x_{-1}, x_0, y_{-1}, y_0 range from zero to infinity and are actual numbers. Here x_n and y_n could be thought of as the densities of two species inside a colony. Then, for each species, we take a, c as the adult survival rate and b, d as the growth rate. Then, in order to see the periodic and chaotic dynamics, bifurcation diagrams are shown. The point of equilibrium for this system is always (0, 0), and it is further represented as E (0, 0). Positive equilibrium solutions are considered in terms of their existence, uniqueness, stability, and attractiveness E $(\bar{x}, \bar{y})$ of (3). The criteria for a 2,7,14-periodic and chaotic solution to equation (3) are also suggested as open challenges.

III. PROPSED MODEL

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Main Results

The uniqueness and existence of an equilibrium solution are guaranteed by the following theorem E ($\bar{x},\ \bar{y}$).

Theorem

1. Suppose

$$0 < a, c < 1 \tag{2}$$

and

$$\alpha = \frac{bd}{(1-a)(1-c)} > 1. \tag{3}$$

A positive equilibrium is achieved in (1). E (($\bar{x},\ \bar{y}$)) such that

$$\frac{\ln \alpha}{1+\frac{d}{1-c}} < \bar{x} < ln\alpha, \frac{ln\alpha}{1+\frac{d}{1-c}} < \bar{y} < ln\alpha \tag{4}$$

and

$$0 < \bar{x} < \frac{b}{1-\alpha}, 0 < \bar{y} < \frac{d}{1-c}. \tag{5}$$

Moreover when

$$0 < \alpha < e^4 \tag{6}$$

There is one specific positive balance E (($\bar{x},\ \bar{y}$)).

Also, when

$$0 < b, d < 1 \tag{7}$$

E (($\bar{x},\ \bar{y}$)) is unique.

2. Suppose

Equation 2 holds and α ≤ 1, then E (0, 0) occupies the singular equilibrium of equation (1).

Proof. 1. Consider the system

$$x = ax + bye^{-y},$$

$$y = dxe^{-x} + cy. \tag{8}$$

We get

$$x(1-a) = bye^{-y}, y(1-c) = dxe^{-x} \tag{9}$$

From (9), we get

$$xy = \frac{bdxye^{-x}e^{-y}}{(1-c)(1-a)}$$

$$\Longrightarrow e^{x+y} = \frac{db}{(1-c)(1-a)}$$

Therefore, since a, c < 1 giving α > 0 we have

$$x + y = lna. \quad (10)$$

Since α > 1, we get x 6= 0 and y 6= 0.

It follows from relations (9) and (10) that

$$y = \frac{lna}{a+\frac{be^{-y}}{(1-a)}}.$$

Similarly, we get,

$$x = \frac{lna}{1+\frac{de^{-x}}{(1-a)}}$$

We consider the function

$$F(x) = x - \frac{lna}{1+\frac{de^{-x}}{(1-c)}} \quad (11)$$

Here $F(0) < 0 \; and \lim_{x\to+\infty} F(x) = +\infty$.

Therefore, a positive equilibrium is present. $\bar{x}$,∈ (0, ∞) and $\bar{y} \in (0, \infty)$

Such that

$$\bar{x} = \frac{lna}{1+\frac{de^{-\bar{x}}}{(1-c)}} \quad (12)$$

And

$$\bar{y} = \frac{lna}{1+\frac{be^{-\bar{y}}}{(1-a)}} \quad (13)$$

Also

$$\bar{x} < \ln a, \bar{y} < lna. \quad (14)$$

Also

$$\frac{1}{1+\frac{de^{-\bar{x}}}{1-c}} > \frac{1}{1+\frac{d}{1-c}}$$

Hence

$$\bar{x} > \frac{\ln a}{1+\frac{d}{1-c}}. \quad (15)$$

Similarly

$$\bar{y} > \frac{\ln a}{1+\frac{b}{1-a}} \quad (16)$$

Therefore, the necessary outcome may be obtained from equations (3.12), (3.13), (3.14), (3.15), and (16).

Since $(\bar{x}, \bar{y})$ satisfies (1), we get (5)

From (11),

$F'(x)1 - \frac{(1-c)}{[d+(1-c)e^x]^2}$

Let $u = (1-c)e^x$

$\Rightarrow F'(x) = 1 - \frac{ud\ ln\ \alpha}{[d+u]^2}$

Now when

$F' = 0, we\ get \frac{ud \ln \alpha}{[d+u]^2} = 1$

i.e.,

$u^2 + u(2d - dln\ \alpha)d^2 = 0.$ (17)

(17) lacks genuine origins during $0 < \alpha < e^4$.

Now when $0 < \alpha < e^4$,

$F'(0) = 1 + \frac{(c-1)d \ln\uparrow\alpha}{(1+d-c)^2} > 0$

$F\ 0\ (x)\ >\ 0\ when\ 0\ <\ \alpha\ <\ e^4$.

We may conclude that a singular positive equilibrium exists $E(\bar{x}, \bar{y})$ when $0 < \alpha < e^4$

Therefore, the outcome.

Also when

$ln\ \alpha < \frac{1-c}{d}$

We get F 0 (x) > 0. Similarly, we are able to $\bar{y}$ is unique when $ln\ \alpha < \frac{1-a}{d}$

$ln\ \alpha < \frac{1-a}{b}$

i.e.,

which holds when $0\ <\ b, d\ <\ 1$.

2. From we get $x\ +\ y\ \leq\ 0$.

Therefore, the non-negative equilibrium of equation (1) is E(0, 0).

Hence the result.

Example Let a = 0.5, c = 0.25, d = 0.85, x−1 = 6.0, y−1 = 9.0, x_0 = 3.0, and y_0 = 2.0.

The scenario shown in Figure 1 occurs when b = 3.3. Here, condition (6) is met since α = 5.2133 < e4. It is clear that a singular equilibrium occurs from Figure 1.

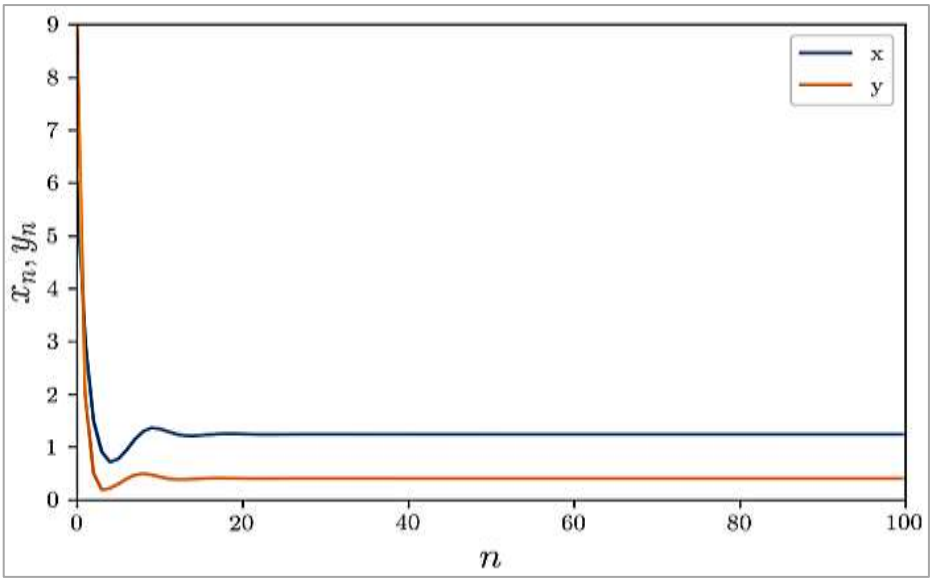


Figure 1: Plots of x_n and y_n of (3.2) for Various b = 2.3 Which Shows the Existence and Uniqueness of The Equilibrium

IV. STABILITY ANALYSIS OF THE SYSTEM

Our research focuses on the nonlinear system's positive solutions, namely on their unique equilibrium, invariance, and attractivity.

$$x_{n+1} = ax_n + by_n e^{-xn-1},$$

$$y_{n+1} = cy_n + dx_n e^{-yn-1} \quad n=0,1, \ldots \tag{18}$$

where a, b, c, d are positive real numbers with 0 < a, c < 1 and the initial conditions x_{-1}, x_0, y_{-1}, y_0 are any positive real numbers chosen at random. Additionally, we examine the point of equilibrium (0,0), which is further represented by E (0,0). In this case, xn and yn represent the respective densities of two species in a particular colony. Next, we determine the growth rate (b) and the survival rate (c) for each breed. Appropriate examples, including figures, are provided to explain the theorems.

In this study we assume

$$0 < a, c < 1 \tag{19}$$

and

$$\alpha = \frac{bd}{(1-a)(1-c)} \tag{20}$$

Main Results

Here we derive the uniqueness and existence of the condition of an equilibrium solution E $(\bar{x}, \bar{y})$.

Theorem 8 If

$$\alpha > 1, \tag{21}$$

then (21) has a positive equilibrium E $(\bar{x}, \bar{y})$. Moreover

$$\frac{1n\,\alpha}{1+\frac{d}{1-c}} < \bar{x} < \frac{1n\,\alpha}{1+\frac{1-c}{d}} \tag{22}$$

and

$$\frac{1n\,\alpha}{1+\frac{b}{1-\alpha}} < \bar{y} < \frac{1n\,\alpha}{1+\frac{1-c}{d}} \quad (23)$$

In particular when

$0 < b, d < 1$ (24)

E $(\bar{x}, \bar{y})$) is unique.

2. If $\alpha \leq 1$, then E (0,0) is the unique equilibrium of (18).

Proof. 1. Consider the system

$x = ax + bye^{-x}$, $y = bxe^{-y}$ + cy. (25)

i.e.

$x(1-\alpha) = bye^{-x}$, $y = (1-c) = bxe^{-y}$ (26)

From (26), we get

$$xy = \frac{bdxye^{-x}e^{-y}}{(1-c)\,(1-a)}$$

$$\Longrightarrow e^{x+y} = \frac{db}{(1-c)(1-a)}$$

Therefore, since a, c < 1 giving $\alpha > 0$ we have

$x + y = \ln\alpha$. (27)

Since $\alpha > 1$, we get $x \neq 0$ and $y \neq 0$.

Relations (26) and (27) imply that

$$y = \frac{ln\,\alpha}{1+\frac{be^{-x}}{(1-\alpha).}}$$

Similarly, we get,

$$x = \frac{ln\,\alpha}{1+\frac{de^{-y}}{(1-c).}}.$$

Again, from relations (26) and (27) we get

$$y = \frac{ln\,\alpha}{1+\frac{(1-c)}{de^{-y}.}}$$

Similarly, we get

$$x = \frac{ln\,\alpha}{1+\frac{(1-\alpha)}{de^{-x}.}}$$

Also, from (26), we have $\frac{x(1-a)(1-c)e^{x}}{b} = dxe^{-y}$, i.e.,

$$\frac{(1-a)(1-c)e^{x}}{b} = de^{-\left[\frac{x(1-a)e^{x}}{b}\right]}.$$

Now consider $F(x) = de^{-\left[\frac{x(1-a)e^{x}}{b}\right]} - \frac{(1-a)(1-c)e^{x}}{b}$.

Here F (0) > 0 and lim $_{x\to+\infty}$ F(x) = −∞.

And F'(x) < 0 when 0 < b, d < 1. Hence, we get a unique positive $\bar{x} \in (0, \infty)$, $\bar{y} \in (0, \infty)$

with the properties

$$\bar{\bar{x}} \frac{ln\ \alpha}{1+\frac{de^{-\bar{y}}}{(1-c)}}, \bar{y} \frac{ln\ \alpha}{1+\frac{de^{-\bar{x}}}{(1-\alpha)}} \quad (28)$$

And

$$\bar{\bar{x}} \frac{ln\ \alpha}{1+\frac{(1-\alpha)}{be^{-\bar{x}}}}, \bar{y} \frac{ln\ \alpha}{1+\frac{(1-c)}{be^{-\bar{y}}}} \quad (29)$$

Also,

$\bar{x}$< ln α, $\bar{y}$ ln α. (30)

Also, $\frac{1}{1+\frac{de^{-\bar{y}}}{1-c}} > \frac{1}{1+\frac{d}{1-c}}$.

Hence,

$$\bar{x} > \frac{ln\ \alpha}{1+\frac{d}{1-c}} \quad (31)$$

Similarly,

$$\bar{y} > \frac{ln\ \alpha}{1+\frac{b}{1-a}} \quad (32)$$

Also, $be^{-\bar{x}} \quad \frac{\ln \alpha}{1+\frac{(1-a)}{}} < \frac{\ln \alpha}{1+\frac{(1-a)}{b}}$.

$$\Longrightarrow \bar{x} < \frac{\ln \alpha}{1+\frac{(1-a)}{b}}. \quad (33)$$

Similarly,

$$\bar{y} > \frac{ln\ \alpha}{1+\frac{(1-c)}{d}} \quad (34)$$

Hence from (28), (29), (30), (31), (32) and (33) the proof is complete.

V. CONCLUSION

In this study, the existence, stability, and boundedness of solutions of second-order difference systems with exponential components have been considered. The analysis shows that the behavior of solutions depends strongly on the coefficients of the system and on the nature of the exponential terms. Under suitable

assumptions, solutions can be shown to exist and to remain bounded as the discrete variable increases. Stability is an important part of the analysis because it describes the response of the system to small changes in the initial conditions. The obtained conditions indicate that appropriate restrictions on the parameters can ensure that such changes do not cause uncontrolled growth of the solutions. Similarly, boundedness conditions help guarantee that the solutions remain within a finite range over time. Thus, the study provides useful criteria for understanding the long-term behavior of the considered second-order systems.

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